# Classical Derivation for Spectral Lines of Hydrogen


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### Contributions

A.R. contributed the classical derivation; C.R. contributed Compton scattering insight; T.N. contributed knowledge of spherical harmonics

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## Abstract

A model for the hydrogen atom is provided. It takes for granted the results of Rutherfords and Comptons experiments, and things that came before them only. The derivation is pathological, here meaning there is a clear definition of cause and effect, and a logical progression from system setup to dimensional analysis. Particles move according to Newtons laws of motion. Energy is from the coulomb potentials. This derivation is classical. It explains the spectral lines of the hydrogen atom and discrete effects.


## Introduction

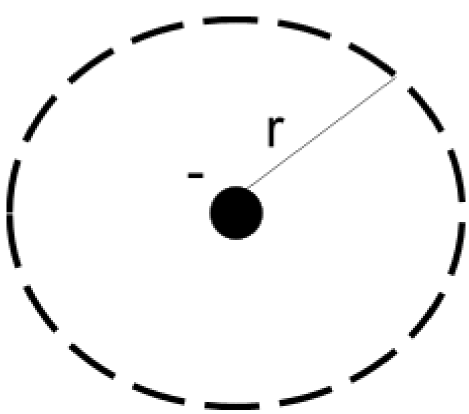
Modeling the hydrogen atom has been of supreme interest and controversy for centuries. The first model described the atom as a rigid sphere. Later the model was updated to include electrons and protons. Later still after Rutherfords experiments the atom was considered to have a heavy, dense nucleus composed of protons. The electrons were considered to be on the periphery of the atom, with the nucleus concentric. Bohr later tried to create a model for the atom with the electrons orbiting the nucleus. Bohrs model contained several aphysical aspects and many unexplained rules. Schrodinger updated the model with the introduction of his equation. Once again there were certain aphysical issues regarding Schrodingers work. The equations were acausal, and there is uncertainty regarding interpretations of the dependent variable. The model contained here is the most physical. There is no dubiousness regarding causality and little to no room left for interpretation. Compton did not provide a model of the atom; however, he performed experiments ultimately giving insight into the interaction of charged particles and photons.

## Results

### The Electron

Compton discovered the radius at which the electron scatters. When considering the electron, there is a sphere of the reduced Compton electron radius, with a charge at its center. The electron will scatter if it's tangent to anything closer than the reduced Compton electron radius.

**Fig. 1: The electron model**



In figure 1, the electron is depicted. The center is the negative charge. The radius is the reduced Compton electron radius. A charge entering the region less than r will trigger scattering. The radius can be explained by the Klein–Nishina formula. The boundary is not inert. It's a semi-rigid manifold. This manifold can be rung like a bell. The ringing of this manifold represents the excitation states of the electron. The ringing is assumed to propagate at the speed of light and be a wave phenomena. This makes the ringing subject to the helmholtz equation.

## The Proton Nucleus

The proton nucleus exerts a force on the electrons' negative charge. This force is described by Coulomb. It is attractive. The classical expression of the force follows

$$F = \frac{ZQ^2}{4\pi\epsilon d^2} \tag{1}$$

Z is the number of positive charges in the nucleus, Q is the elementary charge of electronics and protons, epsilon is the permittivity of free space, d is the distance between the two particles charges. The protons' reduced Compton wavelength is orders of magnitude smaller than the electron one. It's extent in space is much smaller, so we don't have to consider it having an extended region of scattering like the electron.

## The Electron Proton Interaction

The interaction is mediated by Newton's second law. F is force, m is mass, a is acceleration of the electron

$$F = ma \tag{2}$$

Substituting in the electric force yields the following. The proton will be assumed to be stationary, since it's much more massive. Combining 1 and 2 gives.

$$\frac{ZQ^2}{4\pi\epsilon d^2} = ma \tag{3}$$

The electron will follow the electric force towards the proton until the scattering regions are violated. This happens when the distance is less than the reduced Compton scattering radius. Substituting that in and solving for a yields the following.

$$\frac{ZQ^2}{4\pi\epsilon r^2 m} = a \tag{4}$$

At this point in time the electron will scatter. Its velocity's radial component will change sign instantaneously. This doesn't leave the electron unaffected. Being perturbed in such a manner leaves the Compton boundary excited. The electron will follow a parabolic path back down to the proton. This is the solution to Newton's equation of motion. The expression follows. x is the extra distance between the proton and electron

$$x = \frac{1}{2}\frac{ZQ^2 t^2}{4\pi\epsilon r^2 m} \tag{5}$$

The electron will continue to be Compton scattered and Coulomb attracted. If left alone this process might continue forever.

## The Electron Proton Sympathy

Interacting in such a way, the Compton scattering and Coulomb attraction will either become sympathetic or divergent. The frequencies of sympathy are determined by the size and shape of the reduced Compton electron radius. Being a sphere subject to the helmholtz equation, some modes can be investigated. The following is true for an excited sphere

$$j_0(rk) = 0 \tag{6}$$

The excitation is contained in the volume of the sphere, so the boundary equation in 6 can be expressed. j is the spherical Bessel function. k is the wavenumber. r is the boundary of the electron Compton boundary. In more explicit terms the spherical functions for the zeroth order will be the following.

$$\frac{sin(rk)}{rk} = 0 \tag{7}$$

The result in 7 can only be true if the product of the radius and wave number are a multiple of pi, which can be expressed as follows.

$$rk = n\pi \tag{8}$$

Assuming linear dispersion and propagation at the speed of light this equation will give the natural modes of the Compton boundary.

$$\omega = \frac{n\pi c}{r} \tag{9}$$

$$T = \frac{2r}{nc} \tag{10}$$

Expression 10 relates the period of oscillation of the Compton boundary to its radius, the speed of light, and any integer n. To be in sympathy, when the electron comes back down from being scattered the Compton boundary must have the same phase it had when the electron scattered. This puts a constraint on the allowable amplitudes the electron can have beyond the Compton radius. The time in equation 5 is half this period, so a revision to 5 and substitution with 10 follows.

$$A = \frac{1}{4}\frac{1}{2}\frac{ZQ^2T^2}{4\pi\epsilon r^2 m} \tag{11}$$

$$A = \frac{1}{2}\frac{ZQ^2}{4\pi\epsilon mc^2}\frac{1}{n^2} \tag{12}$$

For the electron and proton to be in sympathy the electron's distance to the proton can only be the Compton wavelength plus these amplitudes.

## The Electron Proton Energies

As described in the previous sections the electron and proton form an oscillator. The oscillator can only have certain allowable amplitudes. What's left is to relate those amplitudes to energies. That will be done with the coulomb potential. Imagine the electron proton system is at their maximum distance before the electric force pulls them back together. At this point the system only has potential energy. The amplitude is much smaller than the distance to from the charge centers, so it can be approximated as follows. The work to bring the charges apart a small distance x through the coulomb force can be written.

$$W = \int_0^x F \cdot dl = \int_0^x \frac{ZQ^2 dx}{4\pi\epsilon r^2} = \frac{ZQ^2 x}{4\pi\epsilon r^2} \tag{13}$$

$$E = \frac{ZQ^2 A}{4\pi\epsilon r^2} \tag{14}$$

The potential energy of the maximally separated electron is now related to the amplitude. To get the allowable energies just substitute 12 into 14.

$$E = \frac{ZQ^2}{4\pi\epsilon r^2}\frac{1}{2}\frac{ZQ^2}{4\pi\epsilon mc^2}\frac{1}{n^2} \tag{15}$$

$$E = \frac{1}{2}\left(\frac{ZQ^2}{4\pi\epsilon r}\right)^2\frac{1}{mc^2}\frac{1}{n^2} \tag{16}$$

This is a very curious step in finishing up the energy expression. The hydrogen energy is half the coulomb energy squared divided by the mass energy. The form of the equation is very common in classical derivations. Substituting in the common quantum quantities for variables yields the following

$$E = \frac{Z^2 e^4 m_e}{32\pi^2 \epsilon^2 \hbar^2} \frac{1}{n^2} \tag{17}$$

These are the exact same energy levels as the schrodinger equation solves for.

## Discussion

These results are quite remarkable. The only way the hydrogen atom could be understood before this derivation was through Schrodinger's equation. That equation posed a very serious impediment to anyone who wanted a physical interpretation of the phenomena. There is little to nothing that can be inferred from Schrodinger's equation regarding physical process or causation. It's also impossible to falsify, being a statistics equation primarily. The great triumph of the Schrodinger equation was that it was successful at giving the correct solutions for the spectral lines of the hydrogen atom. Having an undergraduate level classical explanation undermines that unique distinction.

There are a few epistemological quandaries of this model. They are all contained in the evocation of a semirigid shell of Compton wavelength enveloping the electron. I can only refer back to the experiments as credence. There are distinct regions governing the different phenomena observed for the electron. Thomson went as far as declaring he had discovered the electron radius. Later Compton reevaluated Thomson's experiments at higher energy and found a different coefficient, and a larger radius related to the classical by the fine structure constant. Clearly these radii represent some physical boundary, although attributing excitation to them and explicitly giving expression for their excited modes is novel. It's not clear in Schrodinger's equation what is waving, but that problem had to be solved for a logical derivation. The electrons reduced Compton boundary seems like the most likely candidate owing to its size, role, and nature.

## References

These are all more than 100 years old and don't seem to follow the modern publication indexing so I'll just give names.


1. Sir Issac Newton
2. Charles-Augustin de Coulomb
3. Sir Joseph John Thomson
4. Ernest Rutherford
5. Compton, Arthur H. (May 1923). "A Quantum Theory of the Scattering of X-Rays by Light Elements" (PDF). Physical Review. 21 (5): 483–502.


## Ethics declarations

I declare there are no competing interests

## Supplementary Information

Classical physics.